Uniformity Testing over Hypergrids with Subcube Conditioning

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Xi Chen* and Cassandra Marcussen†

*Columbia University
†Harvard University

Outline of this talk

1. **Introduction:** Hypergrids, uniformity testing, subcube conditioning, intractability with standard samples, main result

2. **Previous work:** Conditional sampling models, uniformity testing over hypercubes with subcube conditioning

3. **Our work:** Main result, proof ideas, robust Pisier's inequality over hypergrids, open questions

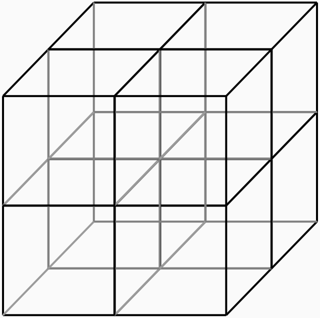
The problem

Uniformity testing over hypergrids

with subcube conditioning.

Hypergrids

We are interested in testing properties of probability distributions supported on a high-dimensional **hypergrid** $[m_1] \times [m_2] \times \cdots \times [m_n]$, where $[m_i] = \{0, 1, \ldots, m_i - 1\}$.



High-dimensional: the dimension n is large.

The problem: Uniformity testing over hypergrids

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2. If $d_{TV}(\mathcal{D},\mathcal{U}) \geq \epsilon$, returns REJECT.

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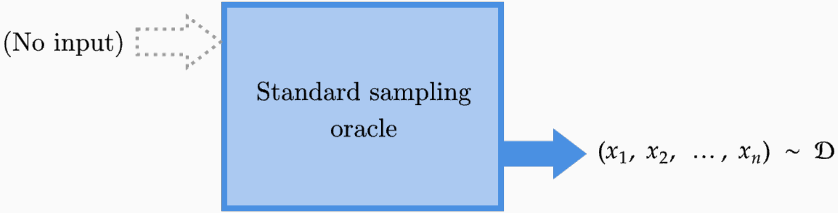

The problem: Uniformity testing in high-dimensional spaces

Intractability with standard samples: Need $\Theta(\sqrt{|\Sigma|}/\epsilon^2)$ samples to test uniformity of a distribution $\mathcal{D}$ supported on Σ. [Pan08, VV14] Bound depends on size of domain, not its structure.

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Two approaches to circumvent this:

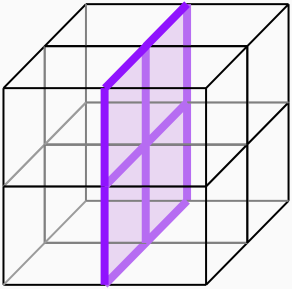
1. Restrict the input distribution to be a product distribution
2. *or* allow stronger sampling access to the distribution.
   This is the focus of our talk.

Stronger access: Subcube conditional oracle model [BC18]

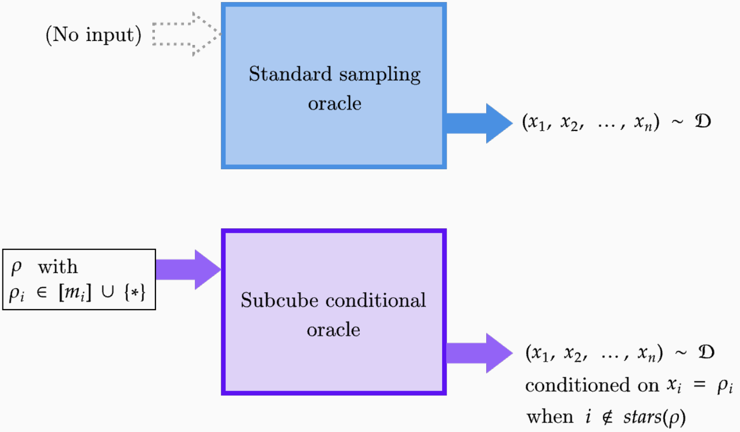
Consider a distribution $\mathcal{D}$ supported on $[m_1] \times [m_2] \times \cdots \times [m_n]$.

Query a **restriction** ρ with $\rho_i \in [m_i] \cup \{*\}$.

The **subcube conditional oracle model** returns a sample $x \sim \mathcal{D}$ conditioned on $x_i = \rho_i$ when $i \notin \text{stars}(\rho)$.



Standard sampling model versus subcube conditional oracle model



The problem

Uniformity testing

- Reasonable starting point in distribution testing

over hypergrids

- Natural structured high-dimensional domain
- Can a “nice” (polynomial) dependency on the alphabet size, dimension, and ϵ be achieved simultaneously?

with subcube conditioning

- Circumvent intractability through stronger sampling access

Main result

Theorem (Chen, M.)
Uniformity testing over hypergrids $[m_1] \times \cdots \times [m_n]$ requires $\tilde{O}(\text{poly}(m)\sqrt{n}/\epsilon^2)$ subcube conditional queries, where $m = \max_{i \in [n]} m_i$.

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Previous work

Classical work: Distribution testing [GR11, BFR+00, BFR+13], property testing on hypergrids [CS13a, CS13b, ADK15, AJMR16, BGKV21]

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Subcube conditioning oracle model: [CRS15, BC18] Adaptation for structured domains such as the hypercube or hypergrid.

- Many applications in learning and testing [CJLW21, KMP23, BCŠV22, BLMT23, CCK+21]
- Uniformity testing over *hypercubes* $\{-1, 1\}^n$ [CCK+21]

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Conditional samples $\geq$ Subcube conditional samples
$\geq$ Standard samples

Testing over hypercubes with subcube conditioning [CCK+21]

Theorem (Canonne, Chen, Kamath, Levi, Waingarten)
Uniformity testing over hypercubes $\{-1, 1\}^n$ requires $\tilde{O}(\sqrt{n}/\epsilon^2)$ subcube conditional queries.

(Compare with $\Omega(\sqrt{n}/\epsilon^2)$ lower bound of [CDKS17, DDK19].)

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High-level ideas of proof: Analyze the distribution under randomly chosen restrictions. Use mean testing as a subroutine.

Mean testing: The mean vector $\mu(\mathcal{D}) \in [-1, 1]^n$ is given by $\mu(\mathcal{D})_i = \mathbb{E}_{x\sim\mathcal{D}}[x_i]$. Mean testing looks at size of $||\mu(\mathcal{D})||_2$.

Random restriction [CCK+21]

A random restriction $\rho \sim \mathcal{R}_\sigma(\mathcal{D})$ is given by the following:

1. (**Additional random step**) Sample a set $S \sim \mathcal{S}_\sigma$ that includes each $i \in [n]$ independently with probability σ.
2. Sample $y \sim \mathcal{D}$.
3. Return the restriction ρ where each ρ_i is set to:

$$\rho_i = \begin{cases} * & \text{if } i \in S \\ y_i & \text{if } i \notin S. \end{cases}$$

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Restriction of a distribution: $(\mathcal{D}_{|\rho})$**:** The distribution of x_S with $x \sim \mathcal{D}$ conditioned on $x_i = \rho_i$ for every $i \notin S$.

Approach in the hypercube case

Lemma:

$$\sum_{j=1}^{\log n} \mathbb{E}_{\rho\sim\mathcal{R}_{\sigma^j}(\mathcal{D})}\left[||\mu(\mathcal{D}_{|\rho})||_2\right] \geq \frac{\tilde{\Omega}(d_{TV}(\mathcal{D},\mathcal{U}))}{\text{poly}(\log n)}$$

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Recursive algorithmic approach:

- Sample random restrictions $\rho \sim \mathcal{R}_\sigma(\mathcal{D})$. If $||\mu(\mathcal{D}_{|\rho})||_2$ is large, reject.
  - Idea: Mean-testing (testing $||\mu(\mathcal{D}_{|\rho})||_2$) is a related but easier problem.
- Otherwise, recurse (take further restrictions on $\mathcal{D}_{|\rho}$).
  - Idea: a typical draw of ρ has large $d_{TV}(\mathcal{D}_{|\rho},\mathcal{U})$ but much smaller dimension.

(Accept if no steps reject.)

Pisier's Inequality

[CCK$^+$21] crucially uses a robust version of Pisier's inequality.

Pisier's inequality connects the ℓ_s norm of a function to differences in the value of the function along the edges of the hypercube.

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Theorem (Pisier's inequality [Pis06])

Let $f : \{-1,1\}^n \to \mathbb{R}$ *be a function with* $\mathbb{E}_x[f(x)] = 0$. *For any* $s \in [1, \infty)$,

$$\left(\mathbb{E}_{x\sim\{\pm1\}^n}\left[|f(x)|^s\right]\right)^{1/s} \leq O(\log n) \cdot \left(\mathbb{E}_{x,y\sim\{\pm1\}^n}\left[\left|\sum_{i\in[n]} y_i x_i L_i f(x)\right|^s\right]\right)^{1/s}.$$

$L_i f(x) = (f(x) - f(x^{(i)}))/2$, where $x^{(i)}$ is the vector obtained from x replacing x_i with $-x_i$.

Use of Pisier's Inequality

Theorem (Pisier's inequality [Pis06]) Let $f : \{-1,1\}^n \to \mathbb{R}$ be a function with $\mathbb{E}_x[f(x)] = 0$.

$$\mathbb{E}_{x\sim\{\pm1\}^n}\left[|f(x)|\right] \le O(\log n) \cdot \mathbb{E}_{x,y\sim\{\pm1\}^n}\left[\left|\sum_{i\in[n]} y_i x_i L_i f(x)\right|\right].$$

$\Uparrow$

Connect to TVD

$\Uparrow$

Connect to differences in probabilities

along edges of the hypercube,

which relates to mean vector.

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Main result

Theorem (Chen, M.)
Uniformity testing over hypergrids $[m_1] \times \cdots \times [m_n]$ requires $\tilde{O}(\mathrm{poly}(m)\sqrt{n}/\epsilon^2)$ subcube conditional queries, where $m = \max_{i \in [n]} m_i$.

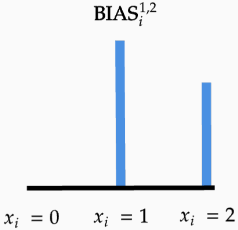
(Compare with $\Omega(\sqrt{nm}/\epsilon^2)$ lower bound of [BGKV21].)

Bias vector

Let $\mathcal{D}$ be a distribution over $[m]^n$, $i \in [n]$ and $c, d \in [m]$. We define

$$\textsc{Bias}_i^{c,d}(\mathcal{D}) = \frac{\Pr_{x\sim\mathcal{D}}[x_i = c] - \Pr_{x\sim\mathcal{D}}[x_i = d]}{\Pr_{x\sim\mathcal{D}}[x_i = c] + \Pr_{x\sim\mathcal{D}}[x_i = d]}.$$

Let $\textsc{Bias}(\mathcal{D})$ denote the bias vector of $\mathcal{D}$, with $n \cdot m^2$ entries $\textsc{Bias}_i^{c,d}(\mathcal{D})$.



Connecting bias vector to total variation distance

Lemma:

$$\sum_{j=1}^{\log n} \mathbb{E}_{\rho \sim \mathcal{R}_{\sigma^j}(\mathcal{D})} \left[||\textsc{Bias}(\mathcal{D}_{|\rho})||_2 \right] \geq \frac{\tilde{\Omega}(d_{TV}(\mathcal{D}, \mathcal{U}))}{\mathrm{poly}(m) \cdot \mathrm{poly}(\log mn)}.$$

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$$\sum_{j=1}^{\log n} \mathbb{E}_{\rho \sim \mathcal{R}_{\sigma^j}(\mathcal{D})} \left[||\textsc{Bias}(\mathcal{D}_{|\rho})||_2 \right] \geq \frac{\tilde{\Omega}(d_{TV}(\mathcal{D}, \mathcal{U}))}{\text{poly}(m) \cdot \text{poly}(\log mn)}.$$

Once proven, can utilize a **recursive algorithmic approach** as well:

- Sample random restrictions $\rho \sim \mathcal{R}_\sigma(\mathcal{D})$. If $||\textsc{Bias}(\mathcal{D}_{|\rho})||_2$ is large, reject.
  - Idea: Can construct a tester for $||\textsc{Bias}(\mathcal{D}_{|\rho})||_2$.
- Otherwise, recurse (take further restrictions on $\mathcal{D}_{|\rho}$).
  - Idea: a typical draw of ρ has large $d_{TV}(\mathcal{D}_{|\rho}, \mathcal{U})$ but much smaller dimension.

(Accept if no steps reject.)

Proving the Inequality

We prove a robust Pisier-style inequality over hypergrids. This allows us to connect (within sub-grids):

Total variation distance to uniform distribution

$\Rightarrow$ Differences in probabilities along edges of the hypergrid

$\Rightarrow$ Bias vector.

Main technical contribution: Pisier's Inequality for Hypergrids

Theorem (Chen, M.)

Let $f : \mathbb{Z}_M \to \mathbb{C}$ *be a function with* $\mathbb{E}_{x\sim\mathbb{Z}_M}[f(x)] = 0$. *For any* $s \in [1, \infty)$,

$$\left(\mathbb{E}_{x\sim\mathbb{Z}_M}\left[\,|f(x)|^s\,\right]\right)^{1/s} \le O(\log n)\cdot\left(\mathbb{E}_{x,y\sim\mathbb{Z}_M}\left[\left|\sum_{i\in[n]} L_i f(x) \sum_{a\in\mathbb{Z}^*_{m_i}} \omega_i^{-ay_i}\omega_i^{ax_i}\right|^s\right]\right)^{1/s}.$$

Notation and terminology:

1. Let $\mathbb{Z}_M$ denote $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_n}$, where $M = (m_1, \ldots, m_n)$. For each $j \in [n]$, let $\mathbb{Z}^*_{m_j} = \{1, \ldots, m_j - 1\}$
2. Let $x^{(i)\to a}$ be the vector obtained from x by replacing x_i with a.
3. *ith coordinate Laplacian operator* [O'D21]:
$$L_i f(x) = f(x) - \mathbb{E}_{a\sim\mathbb{Z}_{m_i}}\left[f(x^{(i)\to a})\right].$$
4. Let $\omega_j = e^{2\pi i/m_j}$ be the primitive m_j-th root of unity.

Use of Pisier's Inequality for Hypergrids

Theorem (Chen, M.)

Let $f : \mathbb{Z}_M \to \mathbb{C}$ be a function with $\mathbb{E}_{x \sim \mathbb{Z}_M}[f(x)] = 0$.

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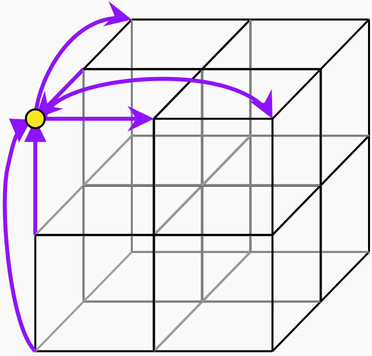
$$L_i f(x) = \sum_{c \in \mathbb{Z}_{m_i}} \frac{f(x) - f(x^{(i) \to c})}{m_i}$$

Robust Pisier's Inequality

We prove a *robust* version of Pisier's inequality over hypergrids.

Robust: For any orientation G of the hypergrid, the RHS of Pisier's inequality only involves x_i for x such that $(x, x^{(i)\to c}) \in G$ for some $c \in \mathbb{Z}_{m_i}$.



Summary

Uniformity testing over hypergrids: accomplished with $\widetilde{O}(\text{poly}(m)\sqrt{n}/\epsilon^2)$ queries to a subcube conditional sampling oracle.

Technical tools

1. **Bias vector:** captures differences in the function's value over edges in the hypergrid.
2. **Robust Pisier's inequality over hypergrids:** an isoperimetric inequality connecting the ℓ_s norm of a function f to its Laplacian operators $L_i f$.

Open Questions

Question 1: Pinning down the complexity of uniformity testing over hypergrids with subcube conditioning as a function of m.

Question 2: Understanding the complexity of identity testing with subcube conditioning, both in the hypercube and hypergrid setting.

- Studied in modified settings [BCŠV22, BCP^{+}23, KMP23].
- Lower bounds.

Question 3: Exploring other natural models of conditional sampling in structured high-dimensional settings.

Thank you for listening!

Questions?

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